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Renormalization of the Periodic Scalar Field Theory by Polchinski's Renormalization Group Method

I. Nándori[1),2)], K. Sailer[2)], U. D. Jentschura[1)], and G. Soff[1)]

[1)] Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany

[2)] Department of Theoretical Physics, University of Debrecen, H-4032, Debrecen, Hungary

Abstract The renormalization group (RG) flow for the two-dimensional sine-Gordon model is determined by means of Polchinski's RG equation at next-to-leading order in the derivative expansion. In this work we have two different goals, (i) to consider the renormalization scheme-dependence of Polchinski's method by matching Polchinski's equation with the Wegner-Houghton equation and with the real space RG equations for the two-dimensional dilute Coulomb-gas, (ii) to go beyond the local potential approximation in the gradient expansion in order to clarify the supposed role of the field-dependent wave-function renormalization. The well-known Coleman fixed point of the sine-Gordon model is recovered after linearization, whereas the flow exhibits strong dependence on the choice of the renormalization scheme when non-linear terms are kept. The RG flow is compared to those obtained in the Wegner-Houghton approach and in the dilute gas approximation for the two-dimensional Coulomb-gas.

1 Introduction

During the last three decades the renormalization of the two-dimensional Coulomb-gas or the equivalent sine-Gordon scalar model in dimension $d = 2$ [1] was investigated in a great detail either perturbatively [2–4] or by means of the differential renormalization group (RG) method in momentum or in coordinate space, using a sharp [5, 6] or a smooth cut-off [1, 7–9]. The well-known Kosterlitz-Thouless phase transition of the two-dimensional Coulomb-gas was obtained in the framework of the Kadanoff-Wilson renormalization scheme, using a dilute gas approximation [1, 7, 8]. Several intuitive approaches exist by which one has tried to go beyond the dilute gas approximation [6, 9–11]. However, these attempts to improve the dilute gas approximation do not fit in a systematic scheme.

In this work we tried to go beyond the local potential approximation in the gradient expansion. Our previous work [6] by means of the Wegner-Houghton approach indicated that wave-function renormalization may play an important role in the RG flow modifying it for strong external fields. Furthermore, it showed that above a critical value of the external field the path integral for the UV modes is dominated by a non-trivial saddle-point and a spinodal instability occurs. In order to clarify the supposed role of the wave-function renormalization we determine the RG flow for the periodic scalar field theory in dimension $d = 2$ in this work. We discuss the field-dependent wave-function renormalization in the framework of Polchinski's RG method.

2 The sine-Gordon, X-Y model and the Coulomb-gas

Several different models, like the sine-Gordon, Thirring, and the X-Y planar models belong to the same universality class, namely to that of the two-dimensional Coulomb-gas. The X-Y model with an external magnetic field consists of classical two-component spins where the magnitude of the spin is $|S_x| = 1$ at each site. The model is given by the action:

$$
\begin{aligned}
S &= \frac{1}{T} \sum_{<x,x'>} \boldsymbol{S}_x \cdot \boldsymbol{S}_{x'} + \frac{1}{T} \sum_x \boldsymbol{S}_x \cdot \boldsymbol{h} \\
&= \frac{1}{T} \sum_{<x,x'>} \cos(\theta_x - \theta_{x'}) + \frac{h}{T} \sum_x \cos(\theta_x)
\end{aligned} \tag{1}
$$

with the temperature T, the external field $h = |\boldsymbol{h}|$, and the angle θ of each spin with an arbitrarily chosen direction. In the model there exist topological excitations, called vortices, which interact via Coulomb interaction. The X-Y model can be mapped by means of the Villain-transformation to a Coulomb-gas [1]. Such a mapping is, however, only valid up to irrelevant interaction terms.

The other example, the two-dimensional sine-Gordon model is a one-component scalar field theory with periodic self-interaction, which is defined by the Euclidean action [2]:

$$S = \int \mathrm{d}^2x \left[\frac{1}{2}(\partial\phi)^2 + u\cos(\beta\phi)\right]. \tag{2}$$

with the Fourier-amplitude u and the length of period β. The equivalence between the X-Y model and the lattice regularized compact sine-Gordon model can be shown by expressing (2) in terms of the compact variable $z(x) = \exp[i\beta\phi(x)]$ [1]. This makes the kinetic energy periodic and introduces vortices in the dynamics.

3 Polchinski's RG equation

In Polchinski's RG method [12] the realization of the differential RG transformations is based on a non-linear generalization of the blocking procedure using a smooth momentum cut-off. In the infinitesimal blocking step the field variable $\Phi(x)$ is separated into the slowly oscillating, IR (ϕ) and the fast oscillating, UV ($\tilde{\phi}$) parts, but both fields contain low- and high-frequency modes due to the smoothness of the cut-off. The propagator for the IR component is suppressed by a properly chosen regulator function $K(p^2/k^2)$ at high frequency above the moving momentum scale k. The introduction of the regulator function generates infinitely many vertices with derivatives of the field. Most of these vertices are considered irrelevant and their flow is neglected. In order to determine Polchinski's RG equation for the one-component scalar field theory, we follow here the method first explained in Ref. [13].

Let us start with the partition function for the scalar field Φ at scale k,

$$Z = \int \mathcal{D}[\Phi]\ \exp\left(-S_k[\Phi]\right) = \int \mathcal{D}[\Phi]\ \exp\left(-\frac{1}{2}\Phi G_k^{-1}\Phi - S_k^I[\Phi]\right), \tag{3}$$

where the action is split into the sum of terms representing the free propagation and the interactions, $\frac{1}{2}\,\Phi\, G_k^{-1}\,\Phi = (2\pi)^{-d}\int \mathrm{d}^dp\,\frac{1}{2}\,\Phi_{-p}\,G_k^{-1}(p^2)\,\Phi_p$ and $S_k^I[\phi]$, respectively, where $G_k^{-1}(p^2) = p^2\,K^{-1}(p^2/k^2)$ stands for the regulated inverse propagator. The regulator function $K(z)$ suppresses the high-frequency modes ($|p| \gg k$) and keeps the low-frequency ones ($|p| \ll k$) unchanged due to the limiting behaviors $K(z) \to 0$ for $z \gg 1$ and $K(z) \to 1$ for $z \ll 1$, respectively. Then the field variable and the propagator are split into the sum of IR and UV parts,

$$\Phi = \phi + \tilde{\phi}, \tag{4}$$

and

$$G_k = G_{k-\Delta k} + \tilde{G}_k, \tag{5}$$

where $\tilde{G}_k$ and $G_{k-\Delta k}$ correspond to $\tilde{\phi}$ and ϕ, respectively. The UV propagator is written as $\tilde{G}_k = \Delta k\ \partial_k G_k$ for infinitesimal Δk. Since both ϕ and $\tilde{\phi}$ are non-vanishing for all momenta it seems as if the degrees of freedom were doubled. In order to have the same number of degrees of freedom as before the blocking one may introduce a new dummy field $\tilde{\Phi}$ by inserting a trivial constant factor in the partition function, written as a Gaussian path integral over the new field $\tilde{\Phi}$ with the arbitrarily chosen propagator $G_D(p^2)$,

$$\begin{aligned} Z &= \int \mathcal{D}[\Phi]\; \exp\left(-S_k[\Phi]\right) \\ &= \mathcal{N} \int \mathcal{D}[\tilde{\Phi}]\mathcal{D}[\Phi]\; \exp\left(-\frac{1}{2}\Phi\, G_k^{-1}\, \Phi - \frac{1}{2}\tilde{\Phi}\, \tilde{G}_D^{-1}\, \tilde{\Phi} - S_k^I[\Phi]\right). \end{aligned} \tag{6}$$

The fields ϕ and $\tilde{\phi}$ are defined by a Bogoliubov-Valatin transformation of the fields Φ and $\tilde{\Phi}$ [13] and the partition function becomes

$$Z = \int \mathcal{D}[\tilde{\phi}]\mathcal{D}[\phi]\; \exp\left(-\frac{1}{2}\,\phi\, G_{k-\delta k}^{-1}\, \phi - \frac{1}{2}\,\tilde{\phi}\, \tilde{G}_k^{-1}\, \tilde{\phi} - S_k^I[\phi + \tilde{\phi}]\right). \tag{7}$$

The blocked action is defined by

$$\exp\left(-S_{k-\delta k}^I[\phi]\right) = \int D[\tilde{\phi}]\; \exp\left(-\frac{1}{2}\tilde{\phi}\, \tilde{G}_k^{-1}\, \tilde{\phi} - S_k^I[\phi + \tilde{\phi}]\right). \tag{8}$$

We assume that the saddle point

$$\tilde{\phi}_c = -\tilde{G}_k \frac{\delta S_k^I[\phi + \tilde{\phi}_c]}{\delta \phi} \tag{9}$$

in this functional integral is $\mathcal{O}(\Delta k)$ since $\tilde{G}_k = \mathcal{O}(\Delta k)$. The expansion of the exponent around $\tilde{\phi} = 0$ yields

$$\begin{aligned} S_k^I[\phi] - S_{k-\delta k}^I[\phi] &= \frac{1}{2}\frac{\delta S_k^I[\phi]}{\delta \phi}\, \tilde{G}_k\, \frac{\delta S_k^I[\phi]}{\delta \phi} \\ &\quad -\frac{1}{2}\mathrm{Tr}\ln\left[\tilde{G}_k^{-1} + \frac{\delta^2 S_k^I[\phi]}{\delta\phi\delta\phi}\right] + \mathcal{O}(\Delta^2 k). \end{aligned} \tag{10}$$

Finally we perform the limit $\Delta k \to 0$,

$$\partial_k S_k^I[\phi] = \frac{1}{2}\frac{\delta S_k^I[\phi]}{\delta \phi}\, \partial_k G_k\, \frac{\delta S_k^I[\phi]}{\delta \phi} - \frac{1}{2}\mathrm{Tr}\left[\partial_k G_k \frac{\delta^2 S_k^I[\phi]}{\delta\phi\delta\phi}\right]. \tag{11}$$

One can rewrite (11) for the complete action [14,15] as

$$\partial_k S_k[\phi] = \frac{1}{2}\int_p \partial_k G_k(p^2)\left[\frac{\delta S_k[\phi]}{\delta\phi_{-p}}\frac{\delta S_k[\phi]}{\delta\phi_p} - \frac{\delta^2 S_k[\phi]}{\delta\phi_{-p}\delta\phi_p}\right.$$

$$\left. -2\,\phi_p\, G_k^{-1}(p^2)\frac{\delta S_k[\phi]}{\delta\phi_p}\right]. \tag{12}$$

The closed functional integro-differential equation (11) could only be obtained because the saddle point dominating the path integral of the blocking in (8) is $\mathcal{O}(\Delta k)$. Such a suppression occurs because the saddle point is driven by $S_k^I[\phi+\tilde{\phi}]$ and its amplitude is controlled by the kinetic energy $\tilde{\phi}\tilde{G}_k^{-1}\tilde{\phi}/2$. But it may happen that the kinetic energy is vanishing at a certain momentum scale $k=k_c$ and a spinodal instability shows up. One may think of models where the regulator function also evolves and at given scale $k=k_c$, it starts to develop a singularity for some mode p_c for which $\left.\frac{dK}{dp^2}\right|_{p_c}=\infty$. The saddle point may become strong and an important tree-level renormalization is observed in this case. This phenomenon is not covered by Polchinski's equation and then a systematical search for the saddle point of (8) has to be performed.

4 Periodicity and Polchinski's equation

In order to solve the functional integro-differential equations (11) and (12), one has to project them to a particular functional subspace. It is generally assumed that the blocked action contains only local interactions, and that it can be expanded in the gradient of the field. We retain the terms up to the next-to-leading order,

$$S_k = \frac{1}{2}\int \mathrm{d}^d x\, Z_k(\phi(x))\,\partial_\mu\phi(x)\,\partial^\mu\phi(x) + \int \mathrm{d}^d x V_k(\phi(x)) \tag{13}$$

with $Z_k(\phi)$ and $V_k(\phi)$ being the field-dependent wave-function renormalization and the potential. The interaction part of the action $S_k^I[\phi]$ is defined as

$$S_k^I[\phi] = S_k[\phi] - \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, G_k^{-1}(p^2)\phi_p\phi_{-p}$$

$$= S_k[\phi] - \frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\, p^2 K^{-1}\left(\frac{p^2}{k^2}\right)\phi_p\phi_{-p}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}^d p_1}{(2\pi)^d}\frac{\mathrm{d}^d p_2}{(2\pi)^d}(-p_1 p_2)\,[Z_{-p_1-p_2}(\phi)$$

$$-\frac{1}{2}\left(K^{-1}\left(\frac{p_1^2}{k^2}\right)+K^{-1}\left(\frac{p_2^2}{k^2}\right)\right)\delta_{p_1+p_2}\Bigg]\phi_{p_1}\phi_{p_2}+\int V_k(\phi)\,, \tag{14}$$

with $Z_{-p_1-p_2}(\phi)=\int \mathrm{d}^dx\, Z_k(\phi_x)\, e^{-ix(p_1+p_2)}$. The Polchinski equation reduces to the RG equations for the dimensionless functions $Z_k(\phi_0)$ and $V_k(\phi_0)$ in dimension $d=2$,

$$(2+k\,\partial_k)V_k(\phi_0)=-[V_k^{(1)}(\phi_0)]^2\,K_0'+(Z_k(\phi_0)-K_0^{-1})\,I_1+V_k^{(2)}(\phi_0)\,I_2\,,$$

$$k\,\partial_k Z_k(\phi_0)=-4\,(Z_k(\phi_0)-K_0^{-1})\,V_k^{(2)}(\phi_0)K_0'$$

$$-2\,V_k^{(1)}(\phi_0)\,Z_k^{(1)}(\phi_0)\,K_0'-2\,[V_k^{(2)}(\phi_0)]^2\,K_0''+Z_k^{(2)}(\phi_0)\,I_2\,, \tag{15}$$

with $Z_k^{(n)}(\phi_0)=\partial_{\phi_0}^n Z_k(\phi_0)$, $V_k^{(n)}(\phi_0)=\partial_{\phi_0}^n V_k(\phi_0)$, where $K'\equiv\partial_{\tilde p^2}K(\tilde p^2)$, $K_0'=\partial_{\tilde p^2}K(\tilde p^2)|_{\tilde p^2=0}$, $I_1=(2\pi)^{-2}\int \mathrm{d}^2\tilde p\,\tilde p^2 K'(\tilde p^2)$, $I_2=(2\pi)^{-2}\int \mathrm{d}^2\tilde p\, K'(\tilde p^2)$ and $\tilde p^2=p^2/k^2$. One can find the same RG equations for the potential and for the wave-function renormalization in the literature by setting the anomalous dimension $\eta=0$ in Ref. [14, 15].

We consider the renormalization of the bare action exhibiting the symmetry

$$\phi(x)\to\phi(x)+\Delta, \tag{16}$$

therefore the potential $V_k(\phi)$ and the wave-function renormalization $Z_k(\phi)$ are expected to remain periodic functions of the field with the length of period Δ,

$$Z_k(\phi(x))=Z_k(\phi(x)+\Delta),\qquad V_k(\phi(x))=V_k(\phi(x)+\Delta). \tag{17}$$

We shall furthermore assume that both the kinetic energy and the interaction term of the action are periodic. It can be seen that the Polchinski equation (11) for the interaction part of the action keeps the periodicity of $S_k^I[\phi]$ with the unchanged period Δ,

$$S_{k-\Delta k}^I[\phi]\;=\;S_k^I[\phi]$$

$$+\Delta k\left(-\frac{1}{2}\frac{\delta S_k^I[\phi]}{\delta\phi}\,\partial_k G_k\,\frac{\delta S_k^I[\phi]}{\delta\phi}+\frac{1}{2}\,\mathrm{Tr}\left[\partial_k G_k\frac{\delta^2 S_k^I[\phi]}{\delta\phi\delta\phi}\right]\right), \tag{18}$$

i.e. the energy $V_k(\phi)$ and the wave-function renormalization $Z_k(\phi)$ satisfy (17).

Instead of equation (11), which is valid for the interaction part of the action, one can use the more usual form of the Polchinski equation (12) which is valid for the complete action.

Although the equation (12) could appear to break the periodicity of the action due to the term $2\phi_p G_k^{-1}(p^2)\frac{\delta S_k[\phi]}{\delta\phi_p}$, this is not the case since

$$2\bigg(\phi_p + \Delta\delta_{p,0}\bigg) G_k^{-1}(p^2)\frac{\delta S_k[\phi+\Delta]}{\delta\phi_p} = 2\phi_p G_k^{-1}(p^2)\frac{\delta S_k[\phi]}{\delta\phi_p} \tag{19}$$

owing to the periodic nature of $S_k[\phi]$ and the property $G_k^{-1}(0) = 0$.

5 Linearized solution

Here we consider the linearized Polchinski equations. First, we use them to show that practically no constraints are laid upon the regulator function by requiring the same classification of the scaling operators at the Gaussian fixed point as obtained in the Wegner-Houghton method. Second, we solve the linearized equations for the periodic blocked action and recover the Coleman fixed point.

One can linearize the equations (15) around the UV Gaussian fixed point: $V_k(\phi_0) = V^* + \delta V_k(\phi_0)$ and $Z_k(\phi_0) = Z^* + \delta Z_k(\phi_0)$ with $V^* = 0$ and $Z^* = 1$ (if the anomalous dimension η is introduced, it is set to zero $\eta = 0$). Then the linearized equations are:

$$\begin{aligned}(2 + k\,\partial_k)\delta V_k(\phi_0) &= I_1\,\delta Z_k(\phi_0) + I_2\,\delta V_k^{(2)}(\phi_0), \\ k\,\partial_k \delta Z_k(\phi_0) &= -4(1-K_0^{-1})\,\delta V_k^{(2)}(\phi_0) + I_2\,\delta Z_k^{(2)}(\phi_0).\end{aligned} \tag{20}$$

In order to calculate the integrals I_1 and I_2, first one has to specify the regulator function $K(\tilde{p}^2)$. One of the most important advantages of Polchinski's RG method is the use of the smooth cut-off, which is compatible with the gradient expansion. Therefore it is possible to consider the renormalization of higher derivatives of the field, as well as the wave-function renormalization $Z_k(\phi)$. The price which has to paid, is that the RG equations depend on the particular choice of the regulator function $K(\tilde{p}^2)$. Unfortunately, this ambiguity cannot be easily removed. A rather straightforward constraint on the regulator function $K(\tilde{p}^2)$ is that it should be defined in such a manner, that around the Gaussian fixed point the classification of the scaling operators into relevant and irrelevant ones be the same as that obtained with the Wegner-Houghton method. In order to formulate this requirement, first we rewrite equation (20) for the field-independent wave-function renormalization $\delta Z_k(\phi_0) = \delta z(k)$:

$$\begin{aligned}(2 + k\,\partial_k)\,\delta V_k(\phi_0) &= I_1\,\delta z(k) + I_2\,\delta V_k^{(2)}(\phi_0), \\ k\,\partial_k \delta z(k) &= -4\,(1-K_0^{-1})\,\delta V_k^{(2)}(\phi_0).\end{aligned} \tag{21}$$

Then we can compare equation (21) to the linearized form of the dimensionless Wegner-Houghton equation [6] around the Gaussian fixed point:

$$
\begin{aligned}
(2 + k\,\partial_k)\,\delta V_k(\phi_0) &= -\alpha\,\log\left[1 + (\delta z(k) + \delta V_k^{(2)}(\phi_0))\right] \\
&= -\alpha(\delta z(k) + \delta V_k^{(2)}(\phi_0))\,, \qquad (22)
\end{aligned}
$$

and

$$
k\,\partial_k\,\delta z(k) = 0\,, \qquad (23)
$$

with

$$
\alpha = \frac{\Omega_2}{2(2\pi)^2} = \frac{1}{4\pi}, \qquad (24)
$$

where $\Omega_2 = 2\pi$ is the solid angle in dimension $d = 2$. In order to get rid of the undesirable tree-level contribution in the linearized Polchinski equations, $K_0 = 1$ has to be chosen. Then, both methods give no wave-function renormalization, $z(k) = Z^* = 1$. The comparison of the field-dependent terms on the right hand sides of the first equations of (21) and (22), we find the constraint $I_2 = -\alpha$,

$$
I_2 \;=\; \int \frac{\mathrm{d}^2\tilde{p}}{(2\pi)^2}\,K'(\tilde{p}^2) = \alpha\int_0^\infty \mathrm{d}x\,K'(x) = \alpha\,[K(\infty) - K(0)] = -\alpha, \qquad (25)
$$

that is satisfied by any regulator function disappearing at infinity $K(\infty) = 0$. Since the field independent terms on the right hands side in the equations for the blocked potential depend on the normalization, the value of the constant I_1 remains undefined. There exist infinitely many regulator functions satisfying the above mentioned rather weak constraints, an example is $K(x) = (1 + ax^n)^{-1}$ or $K(x) = \exp(-a\,x^n)$.

The linearized equations (21) do not give an evolution for the field independent part of the wave-function renormalization $z(k) = 1/\beta^2$, therefore we can rescale the field $\phi \to z^{-1/2}(k)\phi = \beta\phi$ and $\Delta = \frac{2\pi}{\beta}$ becomes the length of period in the internal space, which remains unchanged during the RG transformations. We can look for a solution of equation (20) among periodic functions. The potential $\delta V_k(\phi_0)$ and the wave-function renormalization $\delta Z_k(\phi_0)$ are expanded in Fourier-series:

$$
\delta V_k(\phi_0) = \sum_{n=0}^{\infty} u_n(k)\cos(n\beta\phi_0)\,, \qquad \delta Z_k(\phi_0) = \sum_{n=1}^{\infty} z_n(k)\cos(n\beta\phi_0)\,. \qquad (26)
$$

For the sake of simplicity we consider the potential $\delta V_k(\phi_0)$ and the wave-function renormalization $\delta Z_k(\phi_0)$ which possess the Z(2) symmetry, $\phi_0 \leftrightarrow -\phi_0$. The whole scale dependence occurs in the Fourier amplitudes $u_n(k)$ and $z_n(k)$, the 'coupling constants' of the blocked action. The linearized evolution equations for the Fourier-amplitudes read as follows,

$$\begin{aligned}(2 + k\,\partial_k)u_n(k) &= I_1\, z_n(k) - I_2\, u_n(k)\beta^2 n^2, \\ k\,\partial_k z_n(k) &= -I_2\, z_n(k)\beta^2 n^2\end{aligned} \tag{27}$$

for $n \geq 1$, where $I_2 = -\alpha = -1/(4\pi)$ and the actual value of the integral I_1 is not fixed.

For $Z_k(\phi) = z(k)$ independent of the field, the analytic solution

$$u_n(k) = u_{n0}\left(\frac{k}{\Lambda}\right)^{\alpha\beta^2 n^2 - 2}, \text{ for } n \geq 0 \tag{28}$$

exists in two dimensions with the initial values u_{n0} at the UV momentum cut-off Λ. This reproduces the well-known Coleman-fixed point and the phases of the sine-Gordon model. Depending on the choice of β^2 the Fourier amplitude $u_n(k)$ is a relevant ($\beta^2 < 8\pi/n^2$), marginal ($\beta^2 = 8\pi/n^2$) or irrelevant ($\beta^2 > 8\pi/n^2$) coupling constant.

If the wave-function renormalization $Z_k(\phi)$ is field-dependent, then the second equation in (27) can be solved analitically:

$$z_n(k) = z_{n0}\left(\frac{k}{\Lambda}\right)^{\alpha\beta^2 n^2}, \text{ for } n \geq 1. \tag{29}$$

Therefore, every $z_n(k)$ ($n \geq 1$) is irrelevant whatever be the choice of β^2 and the actual value of the integral I_1. This means that wave-function renormalization is irrelevant for both phases of the model in the UV regime, where the linearized equations hold. In this case, the solution for the Fourier amplitude $u_n(k)$ can be found only numerically, although in the IR domain, far from the UV cut-off $k \ll \Lambda$, the scaling of $u_n(k)$ only depends on the choice of β^2, since all the coupling constants $z_n(k)$ are irrelevant. In Fig. 1, we plot the scaling of the Fourier amplitude $u_1(k)$ when $\beta^2 = 16\pi > \beta_c^2$ and the initial value for $z_1(k)$ is positive and the integral I_1 is set to be equal to: $I_1 = I_2 = -\alpha$. The position of the Coleman fixed point and the phase structure is independent of the choice of the regulator fiction, that of the renormalization scheme, while the actual flow depends on it.

6 Comparison to the Coulomb-gas

During the last two decades the real space RG approach for the Coulomb-gas in dimension $d = 2$ has been investigated in great detail. The real space RG equations for the dilute vortex-gas are well-known and their derivation is given in the literature [1,7,8]:

$$a\,\frac{dh(a)}{da} = [2 - \hbar\alpha T(a)]\; h(a), \qquad a\,\frac{dT(a)}{da} = -\hbar\,\pi\, T(a)^2 h(a)^2 \tag{30}$$

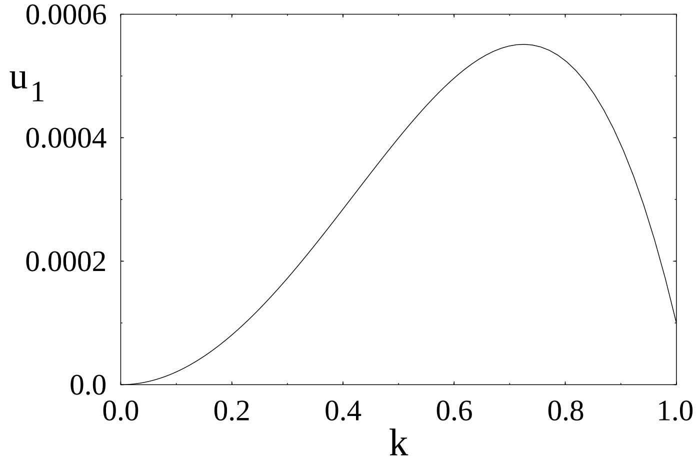

Figure 1: The scaling of the Fourier-amplitude $u_1(k)$ is plotted versus the running momentum cut-off k from $k = 1$ to $k = 0$. This result is obtained by solving the linearized form of Polchinski's RG equation (27) when $\beta^2 = 16\pi$ and the initial value for the Fourier amplitudes are $u_1 = 0.0001$, $z_1 = 0.001\beta^2$ and $u_n = z_n = 0$ if $n > 1$ and the integrals $I_1 = I_2 = -\alpha$.

with the lattice spacing a, and the dimensionless coupling constants h and T. Due to the equivalence between the sine-Gordon model and the two-dimensional vortex or Coulomb-gas, it is possible to compare Polchinski's RG equations for the sine-Gordon model with equation (30). Since the real space RG equations for the Coulomb-gas are non-linear, one has to go beyond the linearized equations (20) in order to compare the equations obtained by the two different methods.

In the real space RG equations (30) for the two-dimensional Coulomb-gas only the field independent wave-function renormalization $Z_k(\phi_0) = z(k) = 1/T = 1/\beta^2$ is taken into account, therefore we consider Polchinski's equations for the following two-dimensional model:

$$S_k[\phi] = \int \mathrm{d}^2 x \left[\frac{1}{2} z(k)(\partial\phi)^2 + u(k)\cos(\phi)\right] \tag{31}$$

where the two coupling constants are the Fourier amplitude $u(k) = u_1(k)$ and the field independent wave-function renormalization $z(k) = z_0(k)$. Inserting the ansatz (31) in Polchinski's equations (15) and neglecting the terms on the right hand sides containing higher harmonics, we find

$$(2 + k\,\partial_k)u(k) = \hbar\,\alpha u(k),$$

$$k\,\partial_k z(k) = -K_0''\,u^2(k). \tag{32}$$

The terms containing the derivatives of the wave-function renormalization with respect to the field ϕ_0 do not appear in (32), since $Z_k(\phi_0) = z(k)$ is independent of the field. In the first equation of (32), the field independent term $z(k)I_1$ and the term $[V_k^{(1)}(\phi_0)]^2 K_0'$ do not give contributions for the Fourier mode $\cos(\phi)$ since $[V_k^{(1)}(\phi_0)]^2 = u^2(k)\sin^2(\phi_0) = u^2(k)(1/2 - 1/2\cos(2\phi_0))$. In the second equation of (32), only $2[V_k^{(2)}(\phi_0)]^2 K_0''$ gives field independent contribution, since $[V_k^{(2)}(\phi_0)]^2 = u^2(k)\cos^2(\phi_0) = u^2(k)(1/2 + 1/2\cos(2\phi_0))$.

These equations should be compared to the flow equations of the sine-Gordon model obtained in the Wegner-Houghton approach [6],

$$(2 + k\,\partial_k)u(k) = \hbar\left[\alpha\frac{u(k)}{z(k)} + \mathcal{O}(u^3)\right],$$

$$k\,\partial_k z(k) = -\hbar\left[\frac{\alpha}{2}\frac{u^2(k)}{z^2(k)} + \mathcal{O}(u^4)\right]. \tag{33}$$

These are Eqs. (7) of [6] rewritten for the dimensionless parameter $u(k)$ and $z(k)$ when the higher order terms on the right hand sides are neglected. The significant difference is that the field independent wave-function renormalization in (32) occurs due to tree-level renormalization. Therefore, the field-independent piece of the wave-function renormalization depends on the 'scheme', which is equivalent to saying that it depends on the details of the blocking procedure (of Polchinski's type with various regulators, or of Wegner–Houghton's type). For the choice $K_0'' = 0$, and on the linear level, the scheme gets closer to the WH approach as to the evolution of the local potential, but the field-independent wave-function renormalization does not alter during the blocking as opposed to the results obtained with the WH method, Eqs. (33). Then the choice $z(k) = 1/T(k) \equiv 1$ is consistent.

Using the equivalence between the lattice regularized compact sine-Gordon model and the X-Y model defined in (1),

$$z = 1/T = 1/\beta^2, \qquad u = h/T\,, \tag{34}$$

one finds that Polchinski's equations (32) can be rewritten in the form of the two-dimensional Coulomb-gas as follows:

$$a\,\frac{dh(a)}{da} = [2 - \hbar\alpha T]\;h, \qquad a\,\frac{dT(a)}{da} = 0 \;\rightarrow\; T(a) = T \equiv 1. \tag{35}$$

These equations (35) are rather different from those obtained by the real space RG approach for the two-dimensional Coulomb-gas (30). At the linear level the RG flow equations are identical irrespectively of the blocking procedure by which they are obtained. The differences of the various approaches occur when the non-linearities are kept that are responsible for the violation of the UV scaling laws.

7 Summary

Our goal was to investigate how far the limitation of the Wegner-Houghton approach due to the truncated gradient expansion can be overcome by using Polchinski's method combined with the gradient expansion. The price one has to pay for introducing the smooth cut-off and not clearly discriminating between UV and IR modes of the quantum fluctuations seems to be high. It has been argued that the application of Polchinski's equations to the renormalization of the scalar field theory with periodicity in the internal space may be troublesome. First, the method mixes the modes above and below the critical scale at which the spinodal instability occurs. Therefore, if such an instability does exist as it is expected from the investigations [6] by means of the Wegner-Houghton method, then the closer we come to this scale, the more questionable the usage of the Polchinski's method is. Second, there is a regulator dependent tree-level renormalization even if the path integral is dominated by a trivial saddle point and no spinodal instability occurs. The latter heavily depends on the choice of the regulator function K, which is a priori not specified by any conditions. Various forms of the cut-off function have been discussed in the literature; the choice was motivated by the need to reproduce the limiting behavior (critical exponents) [14, 15]. Here, we take a different approach and we investigate if rigorous conditions can be established for K based on the matching of Polchinski's equation with the Wegner-Houghton equation. The classification of the scaling operators at the Gaussian fixed point does not imply any substantial contraints on the regulator function, but at least fixes the limiting behavior of K at zero argument, and at infinity, in a unique way. Third, unfortunately, no further conditions can be obtained on the regulator function K by comparing the dilute gas results with those obtained by Polchinski's method for the two-dimensional Coulomb-gas, since the non-linear flow equations are rather different basically due to the regulator-dependent tree-level renormalization in Polchinski's approach.

Summarizing, Polchinski's equation in its linearized form enables one to establish the Coleman fixed point and the phase structure of the two-dimensional sine-Gordon model. Furthermore, in the strong coupling phase $\beta > 8\pi$ the linearization does not loose its validity in the limit $k \to 0$, and we have shown that all couplings associated with the field dependent wave-function renormalization are irrelevant in this phase. They are effectively 'renormalized out' of the theory. For the weak coupling phase the flow equations in their non-linear forms should be solved. Then a strong dependence of the flow on the first and second derivatives of the regulator function at zero momentum occurs, and the various renormalization schemes, Polchinski's, Wegner-Houghton's, and the real space ones give rather different results, although all they were equivalent in the linearized form. Polchinski's scheme gets closer to Wegner-Houghton's one if the derivatives of the regulator function at zero momentum are vanishing. The dependence on the regulator functions, and that on the renormalization scheme (the details of the blocking procedure) modifies the effective (blocked) couplings.

The scheme dependence of the blocked couplings is strongly related to a more general question.

When do we say that the more simple theory reduced by the help of the RG is solved? What is the use of knowing the evolution of the Wilson action? One may be inclined to say that in the $k \to 0$ limit the blocked couplings become physical observables, since only the single mode $p = 0$ is kept. This is, however not true. First, the parameters of the bare (Wilson) action are not observables, just quantities closely related to observables. Second, the dynamics of the 'last' mode is scheme dependent, similarly to the dynamics obtained after eliminating any of the modes during the subsequent blocking steps. This scheme-dependence can only be avoided by using the effective action. The evolution equations for the effective action describe real physics. Their solutions provide the values of the one-particle irreducible (1PI) graphs and the observables are the graphs (transition amplitudes), not the couplings. According to this, the endpoint of these evolution equations is the set of the exact 1PI Green functions.

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